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SOME COMMUTATIVITY THEOREMS FOR SEMI-PRIME RINGS. II

To Professor Kentaro Murata on his sixtieth birthday

YASUYUKI HIRANO, MOTOSHI HONGAN
and HISAO TOMINAGA

Throughout the present paper, $R (\neq 0)$ will represent an associative ring (with or without 1), and C the center of R . Let N be the set of all nilpotent elements of R . We denote by J and D the Jacobson radical and the commutator ideal of R , respectively. A ring R is called s -unital if for each x in R , $x \in Rx \cap xR$. As stated in [9], if R is an s -unital ring, then for any finite subset F of R , there exists an element e in R such that $ex = xe = x$ for all x in F . Such an element e will be called a pseudo-identity of F .

The present objective is to add to the study in the previous paper [8].

1. In this section, we shall prove a commutativity theorem for prime rings which deduces the principal theorem of [1].

Lemma 1. *Let R be a prime ring without non-zero nil ideals in which for every pair of elements x, y there exists a positive integer $n = n(x, y)$ such that $[(xy)^n - (yx)^n, x] = 0$. If C contains a non-zero element c , then for each quasi-regular $a \in R$ and each $x \in R$ there exist positive integers k, l such that $[a, [a, x^k]] = 0$ and $l! [a, x^k]^l = 0$.*

Proof. Using hypothesis for elements $c(1+a)$ and $x(1+a)^{-1}$, there exists a positive integer k such that $c^k[(1+a)x^k(1+a)^{-1} - x^k, c(1+a)] = 0$, where $1+a$ is formally invertible. Then

$$c^{k-1}[a, [a, x^k]] = -c^k[(1+a)x^k(1+a)^{-1} - x^k, c(1+a)](1+a) = 0.$$

Since R is prime, we obtain $[a, [a, x^k]] = 0$. Similarly, $[a, [a, (x^{2k})^m]] = 0$ with some positive integer m . Hence, by [8, Lemma (3)], $0 = {}_{2m}[a, (x^k)^{2m}] = (2m)! [a, x^k]^{2m}$.

Lemma 2. *Let R be a torsion-free prime ring without non-zero nil ideals in which for each pair of elements x, y there exists a positive integer $n = n(x, y)$ such that $[(xy)^n - (yx)^n, x] = 0$. If C contains a non-zero*

element c , then R is a reduced ring.

Proof. Let a be an element of R with $a^2 = 0$. Given $x \in R$, there exists a positive integer n such that $[(ax)^n - (xa)^n, a] = 0$. Then we have $2(ax)^{n+1} = [(ax)^n - (xa)^n, a]x = 0$, and therefore $(ax)^{n+1} = 0$. Hence aR is a nil right ideal. Let $u, v \in aR$. Then, by Lemma 1, there exists a positive integer k such that $[u, [u, (v+c)^k]] = 0$. If $v^2 = 0$ then $kc^{k-1}[u, [u, v]] = [u, [u, (v+c)^k]] = 0$, and so $[u, [u, v]] = 0$. By making use of induction on the nilpotency indices of nilpotent elements v , we can easily see that $[u, [u, v]] = 0$ for all $u, v \in aR$. Thus, [6, Lemma 2.1.1] shows that $a = 0$.

Theorem 1. *Let R be a prime ring without non-zero nil ideals in which for each pair of elements x, y there exists a positive integer $n = n(x, y)$ such that $[(xy)^n - (yx)^n, x] = 0$. If C contains a non-zero element, then R is commutative.*

Proof. First, we show that J is commutative. Let $a, b \in J$. By Lemma 1, there exist then positive integers k, l such that $[a, [a, b^k]] = 0$ and $l! [a, b^k]^l = 0$. If R is of characteristic $p > 0$, then $[a^p, b^k] = pa^{p-1}[a, b^k] = 0$, and hence J is commutative by [7, Theorem]. On the other hand, if R is of characteristic 0, then $[a, b^k] = 0$ by Lemma 2, and hence J is commutative again by [7, Theorem]. Since $D \subseteq J$ by [8, Corollary 1], D is a commutative ideal. Hence, $D \subseteq C$ by [5, Lemma 1.5]. Thus, again by the same lemma, we obtain eventually $R = C$.

Corollary 1 ([1, Theorem 2]). *Let R be a ring without non-zero nil ideals. If for each $x, y \in R$ there exists a positive integer $n = n(x, y)$ such that $(xy)^n - (yx)^n \in C$, then R is commutative.*

Proof. As is well known, R is a subdirect sum of prime rings without non-zero nil ideals. Thus, we may assume that R itself is prime. Then, C is non-zero by [3, Theorem 3], and therefore is commutative by Theorem 1.

2. In this section, we shall prove a commutativity theorem for s -unital semi-prime rings which improves [4, Theorem 1].

Lemma 3. *Suppose that for each $x, y \in R$ there exist positive integers $m = m(x, y)$ and $n = n(x, y)$ such that $[(xy)^n, [(xy)^m, (yx)^m]] = 0$. If $a^2 = 0$ then aR is a nil right ideal.*

Proof. See the proof of [8, Theorem 1].

Lemma 4. *If R is a semi-prime ring satisfying the polynomial identity $[x + y + xy, x + y + yx] = 0$ (or $[xy, yx] = 0$), then R is commutative.*

Proof. Since D is a nil ideal by [2, Theorem 1], D must be 0 by [6, Lemma 2.1.1].

Theorem 2. *Let l, m be fixed positive integers. If R is an s -unital semi-prime ring, then the following are equivalent :*

- 1) R is commutative.
- 2) For each $x, y \in R$ there exists a positive integer $n = n(x, y)$ such that $[x^k y^k - (xy)^k, x] = 0$ and $[x^k y^k - (xy)^k, y] = 0$ ($k = n, n + 1, n + 2$).
- 3) For each $x, y \in R$ there exists a positive integer $n = n(x, y)$ such that $[y^k x^k - (xy)^k, x] = 0$ and $[y^k x^k - (xy)^k, y] = 0$ ($k = n, n + 1, n + 2$).
- 4) For each $x, y \in R$ there exists a positive integer $n = n(x, y)$ such that $[(xy)^k, (yx)^m] = 0$ ($k = n, n + 1$).

Proof. 1) \Rightarrow 2)–4). Trivial.

2) \Rightarrow 1). We first prove the case that R is semi-primitive. Note that 2) is inherited by all subrings and homomorphic images of R . Note also that no complete matrix ring $(S)_t$ over a division ring S ($t > 1$) satisfies 2), as a consideration of $x = E_{12}$ and $y = E_{21}$ shows. Because of these facts and the structure theory of primitive rings, we may assume that R is a division ring. Let x, y be non-zero elements of R . By 2), there exists a positive integer n such that $[x^k y^k - (xy)^k, x] = 0 = [x^k y^k - (xy)^k, y]$ ($k = n, n + 1, n + 2$). Then, we get $[(xy)^k, yx] = 0$ ($k = n, n + 1$). The last equalities yield $[xy, yx] = 0$. Hence R is commutative by Lemma 4.

We now proceed to the general case. Let $x, y \in J$, and e a pseudo-identity of $\{x, y\}$. Then, by making use of the argument employed above, we can easily see that $[(e + x)(e + y), (e + y)(e + x)] = 0$, namely $[x + y + xy, x + y + yx] = 0$. Hence, J is commutative by Lemma 4. Since $D \subseteq J$ by the first step, D is a commutative ideal, and therefore $D \subseteq C$ by [5, Lemma 1.5]. Again by the same lemma, we obtain eventually $R = C$.

3) \Rightarrow 1). The proof is quite similar to that of 2) \Rightarrow 1).

4) \Rightarrow 1). Without loss of generality, we may assume that R is prime. As is easily seen, $[(xy)^n, (yx)^m] = 0$ and $[(xy)^{n+l}, (yx)^m] = 0$ yield

$$(*) \quad (xy)^n [(xy)^l, (yx)^m] = 0.$$

Now, let $a^2 = 0$. Let u, v be arbitrary elements of the nil right ideal aR (Lemma 3), and e a pseudo-identity of $\{u, v\}$. Then, by (*), we readily obtain $[(e+u)(e+v)]^l, [(e+v)(e+u)]^m = 0$. This means that aR is a P. I. ring, and then [6, Lemma 2. 1. 1] shows that $a = 0$; R is a reduced ring. As is well known, the prime reduced ring R has no non-zero zero-divisors. Consequently, again by (*), we get $[(xy)^l, (yx)^m] = 0$. Hence, R is commutative by [8, Theorem 3].

3. Finally, we prove the following commutativity theorem for torsion-free s -unital semi-prime rings.

Theorem 3. *Let R be a torsion-free s -unital semi-prime ring. If for each $x, y \in R$ there exists a positive integer $n = n(x, y)$ such that $[(xy)^n - (yx)^n, x] = 0$, then R is commutative.*

In preparation for the proof of Theorem 3, we establish the following lemmas.

Lemma 5. *Let R be an s -unital ring in which for each pair of elements x, y there exists a positive integer $n = n(x, y)$ such that $[(xy)^n - (yx)^n, x] = 0$. If a is quasi-regular, then for each $x \in R$ there exist positive integers k, l such that $[a, [a, x^k]] = 0$ and $l! [a, x^k]^l = 0$.*

Proof. Let e be a pseudo-identity of $\{a, x\}$, and a' the quasi-inverse of a . Using the hypothesis for the elements $e + a$ and $x(e + a')$, we can find a positive integer k such that $[x^k - (e + a)x^k(e + a'), e + a] = 0$. Then $[a, [a, x^k]] = [x^k - (e + a)x^k(e + a'), e + a](e + a) = 0$, and we can apply the argument employed in the last part of the proof of Lemma 1.

Lemma 6. *Let R be a torsion-free s -unital semi-prime ring in which for each pair of elements x, y there exists a positive integer $n = n(x, y)$ such that $[(xy)^n - (yx)^n, x] = 0$. Then R is a reduced ring.*

Proof. Let $a^2 = 0$. Then aR is a nil right ideal (see the proof of Lemma 2). Let $u, v \in aR$, and e a pseudo-identity of $\{u, v\}$. By Lemma 5, there exists a positive integer k such that $[u, [u, (e+v)^k]] = 0$. If $v^2 = 0$, then $k[u, [u, v]] = [u, [u, (e+v)^k]] = 0$. Hence $[u, [u, v]] = 0$. Now, by making use of induction on the nilpotency indices of nilpotent elements v , we can easily see that $[u, [u, v]] = 0$ for all $u, v \in aR$. Thus, [6, Lemma 2. 1. 1] shows that $a = 0$.

Proof of Theorem 3. In view of Lemmas 5 and 6, we can proceed in

the same way as that of Theorem 1.

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